Chapter 1

Overview of the Lie Transform Method

A basic tool in the theory of normal form for dynamical systems is the Lie transform method which was originally developed in the works of Deprit [21], and extended by Kamel [37], (see also [33, 35, 51]). According to this method, formal normalization transformations of a perturbed system are constructed by means of formal Lie series. The existence of such transformations is provided by the solvability of linear nonhomogeneous equations (involving the Lie derivative along the unperturbed vector field) which are called the homological equations [5]. The “local” traditional approach (see for example [6, 7, 66, 67]) is based on the construction of solutions to homological equations and the corresponding normal forms on domains of local coordinate systems (such as action-angle variables in the Hamiltonian case).

1.1 Setting of the Normalization Problem.

Let $M$ be a smooth manifold and $\mathfrak{X}(M)$ the space of vector fields on $M$. Let $A(\varepsilon, x)$ be an $\varepsilon$-dependent vector field on $M$, that is, a smooth map $A : \mathbb{R} \times M \to TM$ such that $A(\varepsilon, x) \in T_x M$. In other words, the $\varepsilon$-dependent vector field $A$ is a smooth family $\{A_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ of vector fields given by $A_\varepsilon(x) := A(\varepsilon, x)$.

We consider the Taylor expansion of $A_\varepsilon(x)$ at $\varepsilon = 0$

$$A_\varepsilon(x) = A_0(x) + \varepsilon A_1(x) + \cdots + \frac{\varepsilon^k}{k!} A_k(x) + \mathcal{O}(\varepsilon^{k+1}),$$

(1.1.1)

where $A_0, ..., A_k$ are vector fields on $M$ which are defined by the relations

$$\mathcal{L}_{A_s} f = \left. \frac{d^s}{d\varepsilon^s} \right|_{\varepsilon=0} (\mathcal{L}_{A_\varepsilon} f), \quad (s = 1, ..., k),$$

(1.1.2)

for every $f \in C^\infty(M)$. Moreover, $\mathcal{O}(\varepsilon^{k+1})$ denotes an $\varepsilon$-dependent vector field which has zero at $\varepsilon = 0$ of order $k + 1$.

In the context of perturbation theory, for $\varepsilon \ll 1$, we consider the dynamical system of $A_\varepsilon$

$$\frac{dx}{dt} = A_0(x) + \varepsilon A_1(x) + \cdots + \frac{\varepsilon^k}{k!} A_k(x) + \mathcal{O}(\varepsilon^{k+1}),$$

(1.1.3)
which is called the \textit{perturbed system}. The limiting system as $\varepsilon \to 0$
\[
\frac{dx}{dt} = A_0(x)
\]is called the \textit{unperturbed system}. In practice, the unperturbed system usually has some “good” properties in the sense of the integrability theory and symmetries.

\textbf{Definition 1.1.1} An $\varepsilon$-dependent vector field $A_\varepsilon$ on $M$ is said to be in \textbf{normal form of order} $k$ relative to the unperturbed vector field $A_0$ if the perturbation vector fields $A_1,\ldots,A_k$ commute with the unperturbed vector field $A_0$,
\[
\mathcal{L}_{A_0}A_s \equiv [A_0,A_s] = 0 \quad (s = 1,\ldots,k),
\]or, equivalently,
\[
A_s \in \text{ker}(\mathcal{L}_{A_0}) \quad (s = 1,\ldots,k).
\]This normalization approach provides a general setting due to Deprit \cite{22}.

\textbf{Remark 1} A more general definition of normal form of order $k$ of an $\varepsilon$-dependent vector field $A_\varepsilon$ is obtained under replacing condition (1.1.6) by the following
\[
A_s \in \text{ker}(\mathcal{L}_{A_0}^l) \quad (s = 1,\ldots,k),
\]for a certain integer $l \geq 1$ \cite{66}.

\textbf{Definition 1.1.2} Let $N \subseteq M$ be an (nonempty) open domain and $\delta > 0$ a positive number. A smooth mapping $\Phi : (-\delta,\delta) \times N \to M$ is said to be a \textbf{near identity transformation} if for every $\varepsilon \in (-\delta,\delta)$ the map $\Phi_\varepsilon : N \to M$ given by
\[
\Phi_\varepsilon(x) = \Phi(\varepsilon,x)
\]is a diffeomorphism onto its image such and
\[
\Phi_0 = \text{id}.
\]The open subset $N$ is called the domain of definition of the near identity transformation, usually denoted by $\Phi_\varepsilon$. We have the following important property: the pull-back $(\Phi_\varepsilon)^* A_\varepsilon$ of the $\varepsilon$-dependent vector field $A_\varepsilon$ by a near identity transformation $\Phi_\varepsilon$ is again an $\varepsilon$-dependent vector field on $N$ such that
\[
(\Phi_\varepsilon)^* A_\varepsilon|_{\varepsilon=0} = A_0.
\]This means that the near identity transformation $\Phi_\varepsilon$ preserves the unperturbed part of $A_\varepsilon$.

In the case when $N$ is a coordinate chart of $M$ with coordinate functions $x^i : N \to \mathbb{R}$ $(i = 1,\ldots,\dim M)$, condition (1.1.8) can be expressed in the form
\[
x^i \circ \Phi_\varepsilon^{-1} = x^i + O(\varepsilon).
\]Here, the functions $y^i = x^i \circ \Phi_\varepsilon^{-1}$ define a parameter dependent coordinate system on the image $\Phi_\varepsilon(N)$ for every $\varepsilon \in (-\delta,\delta)$. 

Proposition 1.1.1 Let $\Psi : \mathbb{R} \times M \to M$ be a smooth mapping, $(\varepsilon, x) \mapsto \Psi(\varepsilon, x)$ such that $\Psi_0 = \text{id}$. Then, for any open domain $N \subset M$ with compact closure there exists $\delta > 0$, such that for each $\varepsilon \in (\delta, \delta)$ the restriction

$$\Phi_\varepsilon(\cdot) \overset{\text{def}}{=} \Psi(\varepsilon, \cdot)|_N$$  \hspace{1cm} (1.1.10)

is a diffeomorphism onto its image.

Proof. We fix $x \in \bar{N}$. Since $\Psi(0, \cdot) = \text{id}$, $D_x \Psi(0, x)$ is an isomorphism. So, the Implicit Function Theorem implies that there exists a number $\delta_x > 0$, an open neighborhood $W_x$ of $\Psi(0, x) = x$ in $N$, and a unique smooth mapping $g : (-\delta, \delta) \times W_x \to M$ such that for all $(\varepsilon, y) \in (-\delta, \delta) \times W_x$

$$\Psi(\varepsilon, g(\varepsilon, y)) = y.$$  

In other words, for each $\varepsilon \in (-\delta, \delta)$ the mapping $\Phi_\varepsilon$ is a diffeomorphism onto $\Phi_\varepsilon(W_x)$.

Since $\bar{N}$ is compact, it can be covered by a finite number $k$ of neighborhoods $W_{x_1}, W_{x_2}, \ldots, W_{x_k}$. Each one of these neighborhoods has associated a number $\delta_{x_i}$. Let $\delta$ be the minimum of $\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k}$. Then, for each $\varepsilon \in (-\delta, \delta)$, $\Phi_\varepsilon(x)$ is a diffeomorphism onto its image. \qed

Example 1.1.1 Let $M$ be a compact manifold and $Z_\varepsilon$ a smooth $\varepsilon$-dependent (time-dependent) vector field on $M$. Then, the flow $\Phi_\varepsilon = F_{\varepsilon}^{\varepsilon} Z_\varepsilon$ of $Z_\varepsilon$ is a near identity transformation on $N = M$ for all $\varepsilon \in \mathbb{R}$. Conversely, every near identity transformation $\Phi_\varepsilon : M \to M$ can be represented as the flow of the time-dependent vector field

$$Z_\varepsilon(x) = \frac{dF_{\varepsilon}^{\varepsilon} Z_\varepsilon}{d\varepsilon}(F_{\varepsilon}^{\varepsilon} Z_\varepsilon(x)) \quad x \in M.$$  

Example 1.1.2 Let $M = \mathbb{R}^n$ be the Euclidean space and $Z = \sum_{i=1}^{n} Z^i(x) \frac{\partial}{\partial x^i}$ be a vector field on $\mathbb{R}^n$. Then, for any open subset $N \subset \mathbb{R}^n$ with compact closure, there exists $\delta > 0$ such that the mapping

$$x^i \mapsto x^i + \varepsilon Z^i(x)$$

is a near identity transformation with domain of definition $N$, for $\varepsilon \in (-\delta, \delta)$. The inverse of this mapping is of the form

$$x^i \mapsto x^i - \varepsilon Z^i(x) + \mathcal{O}(\varepsilon^2).$$

A more general class of near-identity transformation is described in Proposition 1.1.2. Suppose that for a given $\varepsilon$-dependent vector field $A_\varepsilon$, there exits a near identity transformation $\Phi_\varepsilon$ such that the pull-back $(\Phi_\varepsilon)^* A_\varepsilon$ is in normal form of order $N$,

$$(\Phi_\varepsilon)^* A_\varepsilon = A_0 + \varepsilon A_1 + \cdots + \varepsilon^N A_N + \mathcal{O}(\varepsilon^{N+1}), \quad (1.1.11)$$
[A_0, \tilde{A}_s] = 0 \quad (s = 1, ..., N). \quad (1.1.12)

In this case, \((\Phi_\varepsilon)^* A_\varepsilon\) is called a normalization transformation of order \(N\). Consider the truncated vector field

\[ A_0 + \varepsilon \tilde{A}_N^{(N)} \quad (1.1.13) \]

where

\[ \tilde{A}_N^{(N)} \overset{\text{def}}{=} \tilde{A}_1 + \frac{1}{2} \varepsilon \tilde{A}_2 \cdots + \frac{\varepsilon^{N-1}}{N!} \tilde{A}_N. \quad (1.1.14) \]

Because of (1.1.12) the flow of the truncated vector field can be represented as the composition of the “slow” and “fast” components

\[ \text{Fl}_{A_0 + \tilde{A}_N^{(N)}}^t = \text{Fl}_{\tilde{A}_N^{(N)}}^t \circ \text{Fl}_{A_0}^t. \]

**Long time scale.** To complete this section we recall the following property of the flow of a perturbed vector field

**Proposition 1.1.2** Let \(A_\varepsilon = A_0 + \varepsilon R_\varepsilon\) be an smooth \(\varepsilon\)-dependent vector field. Assume that the unperturbed vector field \(A_0\) is complete on \(M\). Then, for any open domain \(N \subseteq M\) with compact closure and any constant \(\delta > 0\) there is a constant \(L > 0\) such that the flow \(\text{Fl}_{A_\varepsilon}^t\) of \(A_\varepsilon\) is well-defined on \(N\) for all \(t \in [0, \frac{\varepsilon}{\delta}]\) and each \(\varepsilon \in (0, \delta]\).

**Proof.** We will use the following fact which follows from standard properties of flows. The flows of two vector fields \(X\) and \(Y\) on \(M\) are related by

\[ \text{Fl}_X^t \circ \text{Fl}_P^t = \text{Fl}_Y^t. \quad (1.1.15) \]

where \(P_t\) is a time dependent vector field given by

\[ P_t \overset{\text{def}}{=} -X + (\text{Fl}_X^t)^*Y. \quad (1.1.16) \]

Let

\[ (\text{Fl}_{A_0}^t)^* A_\varepsilon - A_0 = \varepsilon R_t(\varepsilon). \quad (1.1.17) \]

where \(R_t(\varepsilon) = (\text{Fl}_{A_0}^t)^* R_\varepsilon\) depends on \(t\) and \(\varepsilon\) smoothly. Fix \(\delta > 0\). The by the Flow Box Theorem and compactness of \(N\) there exists \(L\) such that the flow of \(R_t(\varepsilon)\) is well-defined on \(N\) for \(t \in [0, L]\). Applying formula (1.1.15) for \(X = A_0, Y = A_\varepsilon\) and \(P_t = R_t(\varepsilon)\), we get

\[ \text{Fl}_{A_\varepsilon}^t = \text{Fl}_{A_0}^t \circ \text{Fl}_{R_t(\varepsilon)}^t, \]

and since \(\text{Fl}_{A_0}^t\) is well-defined for all \(t \in \mathbb{R}\), we obtain the desired result. \(\blacksquare\)
1.2 Lie Transforms on Manifolds

The idea of the Lie transform method is searching for a normalization transformation for a perturbed vector field as the flow of a time-dependent vector field where the small parameter $\varepsilon$ plays the role of time. This method allows us to reduce the normalization problem to the study of the solvability of linear nonhomogeneous equations, involving the Lie derivative along the unperturbed vector field $A_0$, which are called the homological equations due to Arnold [5]. Usually, in the context of the formal normalization problem, the derivation of homological equations is given by using the formal Lie series and formal near identity transformations (see, for example [13, 21, 15, 33, 50, 59]). In this Section, we apply the Lie method to construct normal forms (in the sense of Definition 1.1.1) for perturbed systems of general type (which are not necessarily Hamiltonian), which consists of two steps: (1) Taylor expansions of flows and (2) the derivation of homological equations. Our considerations are based on the basic relationship in differential geometry between flows and Lie derivatives and is closed to the approach of Hernard and Roels [34].

We describe three ways for the construction of a near identity transformation $\Phi_\varepsilon$:

(I) Deprit’s version: $\Phi_\varepsilon$ is defined as a flow of a time-dependent vector field $Z_\varepsilon$, where the perturbation parameter $\varepsilon$ plays the role of time variable.

(II) Hori’s version: $\Phi_\varepsilon$ is defined as the time-$\varepsilon$ flow of an autonomous vector field $Z(\varepsilon)$ smoothly depending on the parameter $\varepsilon$.

(III) Generalized version: $\Phi_\varepsilon$ is defined as the time-$\varepsilon$ flow of a time-dependent vector field $Z_\lambda(\varepsilon)$ smoothly depending on the parameter $\lambda$.

1.2.1 Deprit’s method

Let $Z_\varepsilon(x)$ be an smooth $\varepsilon$-dependent (time-dependent) vector field on $M$. For every integer $K \geq 0$, we have the Taylor expansion of $Z_\varepsilon$ at $\varepsilon = 0$:

$$Z_\varepsilon = \sum_{n=0}^{K} \frac{\varepsilon^n}{k!} Z_k + O(\varepsilon^{K+1}). \quad (1.2.1)$$

Let $\Phi_\varepsilon = Fl^\varepsilon_{Z_\varepsilon}$ be the flow of $Z_\varepsilon$,

$$\frac{d Fl^\varepsilon_{Z_\varepsilon}}{d\varepsilon} = Z_\varepsilon \circ Fl^\varepsilon_{Z_\varepsilon}, \quad (1.2.2)$$

$$Fl^0_{Z_\varepsilon} = id. \quad (1.2.3)$$

Assume that there exist an open domain $N \subseteq M$ and $\delta > 0$ such that the flow $Fl^\varepsilon_{Z_\varepsilon}$ is well-defined on $N$ for all $\varepsilon \in (-\delta, \delta)$. In other words, the map $\Phi_\varepsilon \overset{\text{def}}{=} Fl^\varepsilon_{Z_\varepsilon}$ is a near identity transformation with domain of definition $N$. In this case, the time dependent vector field $Z_\varepsilon$ will be called a generator (or generating vector field) of $\Phi_\varepsilon$. 
Suppose we are given an $\varepsilon$-dependent vector field $A_\varepsilon$ on $M$,

$$A_\varepsilon = A_0 + \sum_{k=1}^{K} \varepsilon^k \frac{k!}{k!} A_k + O(\varepsilon^{K+1}). \quad (1.2.4)$$

Consider the pull-back of $A_\varepsilon$ by the flow $\Phi_\varepsilon$ which is an $\varepsilon$-dependent vector field on $N$ with Taylor expansion at $\varepsilon = 0$:

$$\tilde{A}_\varepsilon \overset{\text{def}}{=} (\Phi_\varepsilon)^* A_\varepsilon = A_0 + \sum_{n=1}^{K} \varepsilon^n \frac{n!}{n!} \tilde{A}_n + O(\varepsilon^{K+1}). \quad (1.2.5)$$

The point here is to compute the vector field coefficients $\tilde{A}_n \in \mathfrak{X}(N)$ of this decomposition in terms of the vector fields $Z_n$ and $A_n$ in (1.2.1) and (1.2.4).

By formulas (1.1.2), the coefficients $\tilde{A}_n$ in the Taylor expansion (1.2.5) are given by

$$\tilde{A}_k = \frac{d^k}{d\varepsilon^k} ((\Phi_\varepsilon)^* A_\varepsilon) \Big|_{\varepsilon = 0}, \quad \text{on } N. \quad (1.2.6)$$

We have the following basic formula which describes the relationship between the Lie derivative and the flows of time-dependent vector fields

$$\frac{d}{d\varepsilon} ((\Phi_\varepsilon)^* A_\varepsilon) = (\Phi_\varepsilon)^* \left( L_{Z_\varepsilon} A_\varepsilon + \frac{\partial}{\partial \varepsilon} A_\varepsilon \right), \quad (1.2.7)$$

where $L_{Z_\varepsilon}$ is the Lie derivative along the vector field $L_{Z_\varepsilon}$. Denote by $\mathfrak{X}(\mathbb{R} \times M)$ the space of all $\varepsilon$-dependent vector fields on $M$. Introduce the linear differential operator $\partial_{Z_\varepsilon} : \mathfrak{X}(\mathbb{R} \times M) \to \mathfrak{X}(\mathbb{R} \times M)$ given by

$$\partial_{Z_\varepsilon} \overset{\text{def}}{=} L_{Z_\varepsilon} + \frac{\partial}{\partial \varepsilon}.$$

**Lemma 1.2.1** For every integer $k \geq 1$, the following identity holds

$$\tilde{A}_k = (\partial_{Z_\varepsilon}^k A_\varepsilon) \big|_{\varepsilon = 0}, \quad (1.2.8)$$

where $\partial_{Z_\varepsilon}^k = \partial_{Z_\varepsilon} \circ \ldots \circ \partial_{Z_\varepsilon}$ (k-times).

**Proof.** By formula (1.2.6), we just need to prove that

$$\frac{d^k}{d\varepsilon^k} ((\Phi_\varepsilon)^* A_\varepsilon) = (\Phi_\varepsilon)^* \left( \partial_{Z_\varepsilon}^k A_\varepsilon \right), \quad (1.2.9)$$

for every $k \geq 1$. We proceed by induction. If $k = 1$, equation (1.2.8) coincides with basic formula (1.2.7). Then, we assume that (1.2.9) is true for $k = n - 1$. By direct computation, we get

$$\frac{d^n}{d\varepsilon^n} ((\Phi_\varepsilon)^* A_\varepsilon) = \frac{d}{d\varepsilon} \left( \frac{d^{n-1}}{d\varepsilon^{n-1}} ((\Phi_\varepsilon)^* A_\varepsilon) \right) = \frac{d}{d\varepsilon} ((\Phi_\varepsilon)^* (\partial_{Z_\varepsilon}^{n-1} A_\varepsilon))$$

$$= (\Phi_\varepsilon)^* \partial_{Z_\varepsilon} (\partial_{Z_\varepsilon}^{n-1} A_\varepsilon) = (\Phi_\varepsilon)^* (\partial_{Z_\varepsilon}^n A_\varepsilon).$$
On the other hand, the vector fields $\partial^k_{Z_\varepsilon} A_\varepsilon(x)$ also depend smoothly on $\varepsilon$. Let us suppose that $\partial^k_{Z_\varepsilon} A_\varepsilon(x)$ has the following Taylor expansion at $\varepsilon = 0$:

$$\partial^k_\varepsilon A_\varepsilon = \sum_{m=0}^K \frac{\varepsilon^m}{m!} A^{(k)}_m + O(\varepsilon^{K+1}).$$  \hfill (1.2.10)

Now we prove a result which establishes a recursive relation between the coefficient of the Taylor expansion of vector fields (1.2.10).

**Lemma 1.2.2** The vector fields $A^{(k)}_m \in \mathfrak{X}(M)$ in (1.2.10) satisfy the recurrent relations

$$A^{(k)}_n = A^{(k-1)}_{n+1} + \sum_{m=0}^n C_n^m \mathcal{L} Z_m A^{(k-1)}_{n-m}, \quad \forall \ k \geq 0.$$  \hfill (1.2.11)

Here $C_n^m = \frac{n!}{m!(n-m)!}$.

**Proof.** In order to prove this lemma, we will use the following algebraic fact. For any two linear operators $T$ and $D$ on a vector space, we have the identity

$$[D^n, T] = \sum_{i=0}^{n-1} C_i^n \text{ad}^n_i(T) \cdot D^i,$$  \hfill (1.2.12)

where $\text{ad}_D(T) = [D, T]$. Equation (1.2.10) and formula (1.1.2) implies that

$$A^{(k)}_n = \left\{ \left( \frac{\partial^n}{\partial \varepsilon^n} \circ \partial^k_{Z_\varepsilon} \right) A_\varepsilon \right\}_{\varepsilon = 0}.$$  \hfill (1.2.13)

By direct computation, we obtain

$$\frac{\partial^n}{\partial \varepsilon^n} \circ \partial^k_{Z_\varepsilon} = \frac{\partial^n}{\partial \varepsilon^n} \circ \left( \mathcal{L} Z_\varepsilon + \frac{\partial}{\partial \varepsilon} \right) = \left( \mathcal{L} Z_\varepsilon + \frac{\partial}{\partial \varepsilon} \right) \circ \frac{\partial^n}{\partial \varepsilon^n} + \left[ \frac{\partial^n}{\partial \varepsilon^n}, \partial^k_{Z_\varepsilon} \right],$$

$$= \mathcal{L} Z_\varepsilon \circ \frac{\partial^n}{\partial \varepsilon^n} + \frac{\partial^{n+1}}{\partial \varepsilon^{n+1}} + \left[ \frac{\partial^n}{\partial \varepsilon^n}, \mathcal{L} Z_\varepsilon \right].$$

Taking into account that $[\frac{\partial^n}{\partial \varepsilon^n}, \mathcal{L} Z_\varepsilon] = \mathcal{L} \frac{\partial^n}{\partial \varepsilon^n} Z_\varepsilon$, and applying formula (1.2.12) to operators $D = \frac{\partial^n}{\partial \varepsilon^n}$ and $T = \mathcal{L} Z_\varepsilon$, we obtain

$$\left[ \frac{\partial^n}{\partial \varepsilon^n}, \mathcal{L} Z_\varepsilon \right] = \sum_{k=0}^{n-1} C_k^n \mathcal{L} \frac{\partial^{n-k}}{\partial \varepsilon^{n-k}} Z_\varepsilon \frac{\partial^k}{\partial \varepsilon^k}.$$  \hfill (1.2.14)

Thus, we have

$$\frac{\partial^n}{\partial \varepsilon^n} \circ \partial^k_{Z_\varepsilon} = \frac{\partial^{n+1}}{\partial \varepsilon^{n+1}} + \sum_{k=0}^n C_k^n \mathcal{L} \frac{\partial^{n-k}}{\partial \varepsilon^{n-k}} Z_\varepsilon \frac{\partial^k}{\partial \varepsilon^k}.$$  \hfill (1.2.14)
Applying (1.2.14) to vector field $A_{\varepsilon}$ and evaluating at $\varepsilon = 0$ we obtain (1.2.11). The recursive formula (1.2.11) can be illustrated in Deprit’s triangle, shown in figure 1.1, [21, 50].

Given an $\varepsilon$-dependent vector field $A_{\varepsilon}$, the coefficients of its Taylor expansion (1.2.4) are located in the first column of the Deprit’s triangle. We suppose that we have already calculated the terms of the first (k-1) rows and we want to compute the terms of the $k$-th row. We start with the computation of $A_{k-1}^{(1)}$. This computation involves only the terms on the first column which are above $A_{k-1}$ (i.e., see formula (1.2.11)). Next, we compute $A_{k-2}^{(2)}$ using the term of the second column above $A_{k-1}^{(1)}$.

We can continue with the computations of the terms of the $k$-th rows using formula (1.2.11).

We observe that the coefficients in the Taylor expansion of vector field $\tilde{A}_{\varepsilon}$ (1.2.5) are in the diagonal of Deprit’s triangle. That is, $\tilde{A}_{k} = A_{0}^{(k)}$. It follows by formula (1.2.11) that

$$\tilde{A}_{k} = A_{0}^{(k)} = A_{1}^{(k-1)} + \mathcal{L}_{Z_0}A_{0}^{(k-1)}.$$ 

Finally, we can derive formulas for vector fields $\tilde{A}_{k}$ in terms of the coefficients of Taylor expansion of vector fields $A_{\varepsilon}$ and $Z_{\varepsilon}$.

**Proposition 1.2.3 ([21])** The coefficients $\tilde{A}_{k}$ are given by the formulas

$$\tilde{A}_{k} = A_{k} + \mathcal{L}_{Z_{k-1}}A_{0} + R_{k-1}^{D},$$

for $k = 1, 2, \ldots$, where the vector fields $R_{k-1}^{D} = R_{k-1}^{D}\{Z_0, \ldots, Z_{k-2}; A_0, \ldots, A_{k-1}\}$ are determined in terms of vector fields $Z_0, \ldots, Z_{k-2}$ and $A_0, \ldots, A_{k-1}$ by mean of recursive formulas (1.2.11).

**Proof.** We just need to prove that

$$A_{n}^{(k)} = A_{n+k} + \mathcal{L}_{Z_{n+k-1}}A_{0} + S_{n,k}^{D}.$$
where the vector fields $S_{n,k}^D = S_{n,k}^D \{Z_0, Z_1, \ldots, Z_{n+k-2}; A_0, A_1, \ldots, A_{n+k-1} \}$ are determined in terms of vector fields $Z_0, \ldots, Z_{n+k-2}$ and $A_0, \ldots, A_{n+k-1}$ by means of (1.2.11) for every non-negative integers $k \geq 1, n$. We proceed by induction over $k$. For $k = 1$, formula (1.2.11) reduces to

$$A_n^{(1)} = A_{n+1} + \sum_{m=0}^{n} C_n^m \mathcal{L}_{Z_m} A_{n-m} = A_{n+1} + \mathcal{L}_{Z_n} A_0 + \sum_{m=0}^{n-1} A_0 C_m^n \mathcal{L}_{Z_m} A_{n-m}. $$

Hence, we have $S_{n,1}^D \{Z_0, Z_1, \ldots, Z_n; A_0, A_1, \ldots, A_{n+1} \} = \sum_{m=0}^{n} C_n^m \mathcal{L}_{Z_m} A_{n-m}$. Now, we assume that (1.2.16) hold for $k = d$ and all integer $n$, that is

$$A_n^{(d)} = A_{n+d} + \mathcal{L}_{Z_{n+d-1}} A_0 + S_{n,d}^D.$$  

(1.2.17)

Formula (1.2.11) gives $A_n^{(d+1)} = A_{n+1} + \sum_{m=0}^{n} C_n^m \mathcal{L}_{Z_m} A_{n-m}^{(d)}$. Since the vector fields $A_n^{(d)}$ are given by (1.2.17) for all $n$, we have

$$A_n^{(d+1)} = A_{n+d+1} + \mathcal{L}_{Z_{n+d}} A_0 + S_{n,d}^D + \sum_{m=0}^{n} C_n^m \mathcal{L}_{Z_m} (A_{n+d-m} + \mathcal{L}_{Z_{n+d-m-1}} A_0 + S_{n-m,d}^D).$$

Taking

$$S_{n,d+1}^D \overset{\text{def}}{=} S_{n,d}^D + \sum_{m=0}^{n} C_n^m \mathcal{L}_{Z_m} (A_{n+d-m} + \mathcal{L}_{Z_{n+d-m-1}} A_0 + S_{n-m,d}^D),$$

(1.2.18)

we have that (1.2.16) also hold for $k = d + 1$ and for all $n$. Finally, $\tilde{A}_k = A_0^{(k)}$ and equations (1.2.16) reduce to (1.2.15), where $R_{k-1}^D = S_{0,k}^D$.

As illustration of the recursive formulas (1.2.15), we compute some vector fields $\tilde{A}_k$.

First order:

$$\tilde{A}_1 = \mathcal{L}_{Z_0} A_0 + A_1, \quad \text{and} \quad R_0^D = 0.$$  

Second order:

$$\tilde{A}_2 = A_2 + L_{Z_1} A_0 + R_1^D, \quad \text{and} \quad R_1^D = \mathcal{L}_{Z_0}^2 A_0 + 2 \mathcal{L}_{Z_0} A_1.$$

Third order:

$$\tilde{A}_3 = A_3 + \mathcal{L}_{Z_2} A_0 + R_2^D,$$

$$R_2^D = 3 \mathcal{L}_{Z_0} A_2 + 3 \mathcal{L}_{Z_0}^2 A_1 + \mathcal{L}_{Z_0}^3 A_0 + (2 \mathcal{L}_{Z_0} L_{Z_1} + \mathcal{L}_{Z_1} \mathcal{L}_{Z_0}) A_0 + 3 \mathcal{L}_{Z_1} A_1.$$

In summary, if $\Phi$ is the flow of the vector field $Z_\varepsilon$ (1.2.1) then, the coefficients of the Taylor expansion at $\varepsilon = 0$ of $\tilde{A}_k = \Phi_{\varepsilon}^* A_\varepsilon = A_0 + \varepsilon \tilde{A}_1 + \frac{1}{2!} \varepsilon^2 \tilde{A}_2 + \frac{1}{3!} \varepsilon^3 \tilde{A}_3 + O(\varepsilon^4)$, are given by
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We remark that these formulas remain true if we replace the vector field \( A \) by any \( \varepsilon \)-dependent tensor field on \( M \). In particular, for an \( \varepsilon \)-dependent function \( H_\varepsilon = H_0 + \varepsilon H_1 + \frac{1}{2} \varepsilon^2 H_2 \) we have

\[
H_\varepsilon \circ \Phi_\varepsilon = H_0 + \varepsilon (\mathcal{L}_{Z_0} H_0 + H_1) + \frac{\varepsilon^2}{2} (\mathcal{L}^2_{Z_0} H_0 + 2\mathcal{L}_{Z_0} H_1 + \mathcal{L}_{Z_1} H_0 + H_2) + \mathcal{O}(\varepsilon^3).
\]

1.2.2 Hori’s method

Let \( Z(\varepsilon) \) be an \( \varepsilon \)-dependent vector field on a manifold \( M \) with Taylor expansion at \( \varepsilon = 0 \)

\[
Z(\varepsilon) = \sum_{n=0}^{K} \varepsilon^{k} Z_{k} + \mathcal{O}(\varepsilon^{K+1}). \tag{1.2.19}
\]

We consider \( Z(\varepsilon) \) as an autonomous vector field on \( M \) smoothly depending on the parameter \( \varepsilon \), let \( F^\lambda_{Z(\varepsilon)} \) be the time-\( \lambda \) flow of \( Z(\varepsilon) \),

\[
\begin{align*}
\frac{d F^\lambda_{Z(\varepsilon)}}{d\lambda} & = Z(\varepsilon) \circ F^\lambda_{Z(\varepsilon)}, \tag{1.2.20} \\
F^0_{Z(\varepsilon)} & = \text{id}. \tag{1.2.21}
\end{align*}
\]

We define the family of diffeomorphisms

\[
\Phi_\varepsilon \overset{\text{def}}{=} F^\lambda_{Z(\varepsilon)} \big|_{\lambda=\varepsilon}. \tag{1.2.22}
\]

It is clear that \( \Phi_0 = \text{id} \). Therefore, the mapping \( \Phi_\varepsilon \) (1.2.22) is a near identity transformation which is called Hori’s transformation.

Assume that we are given an \( \varepsilon \)-dependent vector field \( A_\varepsilon \) on \( M \)

\[
A(\varepsilon) = \sum_{n=0}^{K} \varepsilon^{k} A_{k} + \mathcal{O}(\varepsilon^{K+1}), \tag{1.2.23}
\]

and \( \Phi_\varepsilon \) is a Hori’s transformation well-defined on an open subset \( N \subset M \) and generated by the family of autonomous vector fields \( Z(\varepsilon) \) given by (1.2.19). We define the \( \varepsilon \)-dependent vector field \( \tilde{A}_\varepsilon \) by

\[
\tilde{A}_\varepsilon := \Phi_\varepsilon^* A_\varepsilon = A_0 + \sum_{n=1}^{K} A_n + O(\varepsilon^{K+1}). \tag{1.2.24}
\]
Our goal is to get an expression for the vector fields $\tilde{A}_n$ of the decomposition above in terms of the coefficients $Z_n$ and $A_n$. For every fixed $\varepsilon$, formulas (1.1.2) and (1.2.7) imply the following decomposition of vector field $(F_{\varepsilon}^\lambda)^* A(\varepsilon)$ at $\lambda = 0$

\[
(F_{\varepsilon}^\lambda)^* A(\varepsilon) = \sum_{m=0}^{K} \frac{\lambda^m}{m!} L_{Z(\varepsilon)}^m A(\varepsilon).
\] (1.2.25)

Putting $\lambda = \varepsilon$ into (1.2.25) and using the Taylor expansion (1.2.23) of $A(\varepsilon)$, we get

\[
\Phi_{\varepsilon}^* A_{\varepsilon} = \sum_{n=0}^{K} \frac{\varepsilon^n}{n!} \sum_{m=0}^{n} C_{m}^n L_{Z(\varepsilon)}^m A_{n-m} + O(\varepsilon^{K+1}).
\]

In terms of the coefficients of Taylor decomposition of $Z(\varepsilon)$, the Lie derivative operator $L_{Z(\varepsilon)}^m$ takes the form

\[
L_{Z(\varepsilon)}^m = (L_{Z_{0}} + \varepsilon L_{Z_{1}} + \frac{\varepsilon^2}{2!} L_{Z_{2}} + \frac{\varepsilon^3}{3!} L_{Z_{3}} + \ldots)^m.
\]

The Lie operators $L_{Z(\varepsilon)}^m$ depend smoothly on $\varepsilon$. So, we have the following decomposition

\[
L_{Z(\varepsilon)}^m = \sum_{j=0}^{m} \frac{\varepsilon^j}{j!} \hat{L}_{j}^{(m)},
\] (1.2.26)

where the differential operators $\hat{L}_{j}^{(m)}$ are defined by the recurrent relations

\[
\hat{L}_{j}^{(m)} = \sum_{i=0}^{j} C_{i}^{k} L_{Z_{i}} \circ \hat{L}_{j-i}^{(m-1)}
\] (1.2.27)

with

\[
\hat{L}_{0}^{(0)} = \text{id}, \quad \hat{L}_{0}^{(m)} \equiv 0,
\]

\[
\hat{L}_{j}^{(1)} = L_{Z_{j}}, \quad \text{and} \quad \hat{L}_{0}^{(m)} = L_{Z_{0}}^{m}.
\]

**Proposition 1.2.4** Vector fields $\tilde{A}_n$ in (1.2.24) are given by the recursive formulas

\[
\tilde{A}_n = \sum_{m=0}^{n} \sum_{i=0}^{n-m} C_{m}^{n} C_{m}^{n-m-i} \hat{L}_{j-i}^{(m)} A_{n-m-i}.
\] (1.2.28)

**Proof.** By (1.1.2), we have $\tilde{A}_n = \frac{d^n}{d\varepsilon^n} \mid_{\varepsilon=0} (\Phi_{\varepsilon}^* A_{\varepsilon})$. By direct computation, we obtain

\[
\frac{d^n}{d\varepsilon^n} (\Phi_{\varepsilon}^* A_{\varepsilon}) = \sum_{m=0}^{K} \frac{d^n}{d\varepsilon^n} \left( \frac{\varepsilon^m}{m!} L_{Z(\varepsilon)}^m A_{\varepsilon} \right) = \sum_{m=0}^{K} \sum_{i=0}^{n} C_{m}^{n} \frac{\varepsilon^{m+i-n}}{(m+i-n)!} \frac{d^i}{d\varepsilon^i} \left( L_{Z(\varepsilon)}^m A_{\varepsilon} \right).
\]

Thus

\[
\tilde{A}_n = \sum_{m=0}^{n} C_{m}^{n-m} \frac{d^{n-m}}{d\varepsilon^{n-m}} \mid_{\varepsilon=0} \left( L_{Z(\varepsilon)}^m A_{\varepsilon} \right).
\]
By (1.2.23) and (1.2.26), we have

\[ \mathcal{L}_{Z(\varepsilon)}^m A_{\varepsilon} = \sum_{k=0}^{K} \sum_{m=0}^{K} \frac{\varepsilon^k}{k!} \sum_{i=0}^{k} C_{i}^{k,m} A_{m-i}. \]

Hence,

\[ \frac{d^{n-m}}{d\varepsilon^{n-m}} \bigg|_{\varepsilon=0} \left( \mathcal{L}_{Z(\varepsilon)}^m A_{\varepsilon} \right) = \sum_{i=0}^{n-m} C_{i}^{n-m,m} A_{n-m-i}. \]

Therefore, we have

\[ \bar{A}_{n} = \sum_{m=0}^{n} \sum_{i=0}^{n-m} C_{i}^{n,m} A_{n-m-i}. \]

Analogously to Deprit’s method, formulas (1.2.28) of vector fields \( \bar{A}_{k} \) (1.2.24) can be written as

\[ \bar{A}_{k} = A_{k} + k \mathcal{L}_{Z_{k-1}} A_{0} + R_{k-1}^{H}, \]

where the vector fields \( R_{k-1}^{H} = R_{k-1}^{H} \{ Z_{0}, ..., Z_{k-2}, A_{0}, ..., A_{k-1} \} \) are determined in terms of vector fields \( Z_{0}, ..., Z_{k-2} \) and \( A_{0}, ..., A_{k-1} \) by means of recurrent formulas (1.2.27) and (1.2.28). Therefore, if \( \Phi_{\varepsilon} \) is the near transformation (1.2.22), then the coefficients of the Taylor expansion at \( \varepsilon = 0 \) of \( \bar{A}_{\varepsilon} = \Phi_{\varepsilon}^{*} A_{\varepsilon} = A_{0} + \varepsilon \bar{A}_{1} + \frac{1}{2!} \varepsilon^{2} \bar{A}_{2} + \frac{1}{3!} \varepsilon^{3} \bar{A}_{3} + O(\varepsilon^{4}) \), are given by

<table>
<thead>
<tr>
<th>( \bar{A}_{1} )</th>
<th>( \mathcal{L}<em>{Z</em>{0}} A_{0} + A_{1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{A}_{2} )</td>
<td>( \mathcal{L}<em>{Z</em>{0}}^{2} A_{0} + 2 \mathcal{L}<em>{Z</em>{0}} A_{1} + 2 \mathcal{L}<em>{Z</em>{1}} A_{0} + A_{2} )</td>
</tr>
<tr>
<td>( \bar{A}_{3} )</td>
<td>( 3 \mathcal{L}<em>{Z</em>{0}} A_{2} + 3 \mathcal{L}<em>{Z</em>{0}}^{2} A_{1} + \mathcal{L}<em>{Z</em>{0}}^{3} A_{0} + 3 \mathcal{L}<em>{Z</em>{0}} \mathcal{L}<em>{Z</em>{1}} A_{0} + 3 \mathcal{L}<em>{Z</em>{1}} A_{0} + A_{2} + 3 \mathcal{L}<em>{Z</em>{1}} \mathcal{L}<em>{Z</em>{0}} A_{0} + 6 \mathcal{L}<em>{Z</em>{1}} A_{1} + 3 \mathcal{L}<em>{Z</em>{2}} A_{0} + A_{3} )</td>
</tr>
</tbody>
</table>

### 1.2.3 Generalized scheme

Now, we suppose that a vector field \( Z_{\lambda}(\varepsilon) \) is given, and it depends smoothly depending on the parameters \( \lambda \) and \( \varepsilon \). Computing the Taylor expansion at \( \lambda = 0 \) and \( \varepsilon = 0 \)

\[ Z_{\lambda}(\varepsilon) = \sum_{k=0}^{K} \sum_{m=0}^{K} \frac{\lambda^{k} \varepsilon^{m}}{k! m!} Z_{k,m} + O(\lambda^{K+1}) + O(\varepsilon^{K+1}), \quad (1.2.29) \]

where \( Z_{k,m} = \left. \frac{\partial^{k+m} Z_{\lambda}(\varepsilon)}{\partial \lambda^{k} \partial \varepsilon^{m}} \right|_{\lambda=0, \varepsilon=0} \). Let \( \text{Fl}_{\lambda}^{\lambda} Z_{\lambda}(\varepsilon) \) be the flow of \( Z_{\lambda}(\varepsilon) \),

\[ \frac{d \text{Fl}_{\lambda}^{\lambda} Z_{\lambda}(\varepsilon)}{d\lambda} = Z_{\lambda}(\varepsilon) \circ \text{Fl}_{\lambda}^{\lambda} Z_{\lambda}(\varepsilon), \]

\[ \text{Fl}_{\lambda}^{\lambda} Z_{\lambda}(\varepsilon) = \text{id}. \]
Define the map
\[ \Phi_{\varepsilon} \overset{\text{def}}{=} \left. \text{Fl}^\lambda_{Z_{\lambda}(\varepsilon)} \right|_{\lambda=\varepsilon} \tag{1.2.30} \]
with \( \Phi_0 = \text{id} \). Assuming that \( \Phi_{\varepsilon} \) is well-defined on an open subset in \( M \) for all sufficiently small \( \varepsilon \), we arrive at the following generalized version formulas of Deprit’s method and Hori’s method.

**Proposition 1.2.5** For any open domain \( N \subseteq M \) with compact closure there exists \( \delta > 0 \) such that for all \( \varepsilon \in (-\delta, \delta) \), the mapping \( \Phi_{\varepsilon} \) in (1.2.30) is well-defined on \( N \) and gives a diffeomorphism from \( N \) onto its image. Moreover, the coefficients of Taylor expansion of third order at \( \varepsilon = 0 \) of the pull-back \( \Phi_{\varepsilon}^* A_{\varepsilon} = A_0 + \varepsilon \tilde{A}_1 + \frac{\varepsilon^2}{2!} \tilde{A}_2 + \frac{\varepsilon^3}{3!} \tilde{A}_3 + O(\varepsilon^4) \) are given by

| \( \tilde{A}_1 \) | \( \mathcal{L}_{Z_{0,0}} A_0 + A_1 \) |
| \( \tilde{A}_2 \) | \( \mathcal{L}^2_{Z_{0,0}} A_0 + 2\mathcal{L}_{Z_{0,0}} A_1 + 2\mathcal{L}_{Z_{1,0}} A_0 + \mathcal{L}_{Z_{1,0}} A_0 + A_2 \) |
| \( \tilde{A}_3 \) | \( 3\mathcal{L}_{Z_{0,0}} A_2 + 3\mathcal{L}^2_{Z_{0,0}} A_1 + 3\mathcal{L}_{Z_{1,0}} A_1 + \mathcal{L}_{Z_{0,0}} A_0 + \mathcal{L}_{Z_{0,1}} A_0 + 6\mathcal{L}_{Z_{0,1}} A_1 + 3\mathcal{L}_{Z_{0,2}} A_0 + \mathcal{L}_{Z_{2,0}} A_0 + 2\mathcal{L}_{Z_{0,0}} \mathcal{L}_{Z_{1,0}} A_0 + \mathcal{L}_{Z_{1,0}} \mathcal{L}_{Z_{0,0}} A_0 + 3\mathcal{L}_{Z_{1,1}} A_0 + A_3 \) |

**Proof.** By the flow box theorem for and compactness argument, there exists a \( \delta > 0 \) such that the flow \( \text{Fl}^\lambda_{Z_{\lambda}(\varepsilon)} \) is well-defined on \( \bar{N} \) for all \( \lambda \in (-\delta, \delta) \) and \( \varepsilon \in [-1, 1] \).

Now, we fix \( \varepsilon \) and consider the following decomposition \( Z_{\lambda}(\varepsilon) = \sum_{k=0}^{K} \frac{\lambda^k}{k!} Z_k(\varepsilon) + O(\lambda^{K+1}) \). Using formulas of Deprit, we obtain

\[
\left( \text{Fl}^\lambda_{Z_{\lambda}(\varepsilon)} \right)^* A(\varepsilon) = A(\varepsilon) + \lambda \mathcal{L}_{Z_{0}(\varepsilon)} A(\varepsilon) + \frac{\lambda^2}{2!} \left( \mathcal{L}_{Z_{1}(\varepsilon)} + \mathcal{L}^2_{Z_{0}(\varepsilon)} \right) A(\varepsilon)
+ \frac{\lambda^3}{6!} \left( \mathcal{L}_{Z_{2}(\varepsilon)} + \mathcal{L}^3_{Z_{0}(\varepsilon)} + 2\mathcal{L}_{Z_{0}(\varepsilon)} \mathcal{L}_{Z_{1}(\varepsilon)} + \mathcal{L}_{Z_{1}(\varepsilon)} \mathcal{L}_{Z_{0}(\varepsilon)} \right) + O(\lambda^4)
\]

Putting \( \lambda = \varepsilon \) and using formulas (1.2.23), (1.2.29), we derive the desired formulas. \( \blacksquare \)

Proposition (1.2.5) gives a general approach of Deprit’s method and Hori’s method, respectively. Indeed,

- in the Deprit case, vector field \( Z_{\lambda}(\varepsilon) = Z_\lambda \) is independent of \( \varepsilon \). Thus, \( Z_{k,m} = 0 \) if \( m \geq 1 \) and formulas of Proposition (1.2.5) coincides with formulas of Deprit’s method;

- in the Hori case, \( Z_{\lambda}(\varepsilon) = Z(\varepsilon) \) is independent of \( \lambda \). It follows that \( Z_{k,m} = 0 \) if \( k \geq 1 \) and formulas of Proposition (1.2.5) reduce to formulas of Hori’s method.
1.2.4 Homological equations

Now, we return to the normalization problem for a given vector field \( A_\varepsilon \). If we suppose that \( A_\varepsilon \) admits a normalization of order \( K \) and that normalization transformation \( \Phi_\varepsilon : N \to M \) given as the flow of a vector field \( Z_\varepsilon = \sum_{i=0}^{K-1} \varepsilon^i Z_i + O(\varepsilon^K) \), we obtain that vector fields \( A_k \) and \( Z_k \) satisfy certain equations called homological equations.

**Proposition 1.2.6** Assume that an \( \varepsilon \)-dependent vector field \( A_\varepsilon \) admits a near identity transformation \( \Phi_\varepsilon : N \to M \) which brings \( A_\varepsilon \) to normal form (1.1.11). (1.1.12) of order \( K \) for a certain vector fields \( \tilde{A}_1, \ldots, \tilde{A}_K \). Then, the coefficients \( Z_0, \ldots, Z_{K-1} \) of the Taylor expansion of \( Z_\varepsilon \) must satisfy the following equations on \( N \):

\[
\mathcal{L}_{A_0} Z_{k-1} = A_k - \tilde{A}_k + R_{k-1}^D \{ Z_0, \ldots, Z_{k-2}; A_0, \ldots, A_{k-1} \}, \quad (1.2.31)
\]

\[
\mathcal{L}_{A_0} \tilde{A}_k = 0, \quad (1.2.32)
\]

for \( k = 1, \ldots, K \). Here, the vector fields \( R_{k-1}^D \) are described in Proposition 1.2.3.

**Proof:** We assume that \( \Phi_\varepsilon \) is well defined on \( N \) and is the normalization transformation of \( A_\varepsilon \) with generating vector field

\[
Z_\varepsilon = Z_0 + \varepsilon Z_1 + \ldots + \frac{\varepsilon^{K-1}}{(K-1)!} Z_{K-1} + O(\varepsilon^K).
\]

So, vector field

\[
\Phi_\varepsilon^* A_\varepsilon = A_0 + \varepsilon \tilde{A}_1 + \ldots + \frac{\varepsilon^K}{K!} \tilde{A}_K + O(\varepsilon^K),
\]

is in normal form of order \( K \) relative to vector field \( A_0 \). That is,

\[
\mathcal{L}_{A_0} \tilde{A}_s = 0, \quad s = 1, 2, \ldots, N.
\]

Furthermore, Proposition 1.2.3 asserts that coefficients \( Z_k \) of Taylor expansion of \( Z_\varepsilon \) satisfy the following equations on \( N \)

\[
\tilde{A}_k = A_k + \mathcal{L}_{Z_{k-1}} A_0 + R_{k-1}^D, \quad (1.2.33)
\]

for \( k = 1, 2, \ldots, K \), where vector fields \( R_{k-1}^D = R_{k-1}^D \{ Z_0, \ldots, Z_{k-2}; A_0, \ldots, A_{k-1} \} \) are determined in terms of vector fields \( Z_0, \ldots, Z_{k-2} \) and \( A_0, \ldots, A_{k-1} \) by mean of recurrent formulas (1.2.11). Taking into account that \( \mathcal{L}_{Z_{K-1}} A_0 = -\mathcal{L}_{A_0} Z_{K-1} \), formulas (1.2.33) are equivalent to (1.2.31).

The converse statement of Proposition above is true.

**Proposition 1.2.7** Assume that there exist an open domain \( N \subseteq M \) and \( \delta > 0 \) such that the following conditions hold

(a) there are vector fields \( Z_0, \ldots, Z_{K-1} \) and \( \tilde{A}_1, \ldots, \tilde{A}_K \) satisfying on \( N \) equations (1.2.31), (1.2.32) on \( N \) for \( k = 1, \ldots, K \).
(b) the flow $F^{\varepsilon}_{Z_{\varepsilon}^{(K)}}$ of the $\varepsilon$-dependent vector field

$$Z_{\varepsilon}^{(K)} = \sum_{n=0}^{K} \frac{\varepsilon^n}{n!}Z_k$$  \hspace{1cm} (1.2.34)

is well defined on $N$ for all $\varepsilon \in (-\delta, \delta)$.

Then, the near identity transformation

$$\Phi_{\varepsilon} = F^{\varepsilon}_{Z_{\varepsilon}^{(K)}}$$  \hspace{1cm} (1.2.35)

brings the $\varepsilon$-dependent vector field $A_{\varepsilon}$ to normal form of order $O(\varepsilon^K)$ on $N$ relative to $A_0$. In particular, if $M$ is compact, then condition (b) holds on $N = M$.

Proof. Let $\tilde{A}_{\varepsilon} := \Phi_{\varepsilon}^*A_{\varepsilon} = A_0 + \varepsilon\tilde{A}_1 + \ldots + \frac{\varepsilon^K}{K!}\tilde{A}_K$. By Proposition 1.2.3, vector fields $\tilde{A}_k$ are given by

$$\tilde{A}_k = A_k + L_{A_0}Z_{k-1}A_0 + R_k^{D}.$$ 

Hence, we have that $\tilde{A}_k = \tilde{A}_k$ for all $k = 1, 2, \ldots, K$. Since, vector fields $\tilde{A}_k$ satisfy equations (1.2.32), the $\varepsilon$-dependent vector field $\tilde{A}_{\varepsilon}$ is in normal form and $\Phi_{\varepsilon}$ is a normalization transformation.

If $M$ is compact then the unperturbed vector field $A_0$ is complete. Proposition 1.1.2 implies that the near identity transformations $\Phi_{\varepsilon}$ is well defined in any open domain $N \subset M$ with compact closure. Since $M$ is compact, we have $\tilde{M} = M$. So, $\Phi_{\varepsilon}$ is well defined in $N = M$.

Proposition 1.2.7 states that the normalization problem for a given $\varepsilon$-dependent vector $A_{\varepsilon}$ depends on the solvability of equations (1.2.31),(1.2.32). It means that if we want to find an infinitesimal generator (1.2.34) of normalization transformation (1.2.35) then we have to solve , on several steps, the equations for vector fields $Z$ and $\tilde{W}$ on $N$ of the form

$$L_{A_0}Z = W - \tilde{W},$$  \hspace{1cm} (1.2.36)

$$L_{A_0}\tilde{W} = 0,$$  \hspace{1cm} (1.2.37)

where $W$ is a given vector field.

Indeed, on the first step we have to find vector fields $Z_0$ and $\tilde{A}_1$ satisfying the equations

$$L_{A_0}Z_0 = A_1 - \tilde{A}_1,$$  \hspace{1cm} (1.2.38)

$$L_{A_0}\tilde{A}_1 = 0.$$ 

On the second step, we need to find the vector fields $Z_1$ and $\tilde{A}_1$ satisfying the equations

$$L_{A_0}Z_1 = A_2 - \tilde{A}_2 + L_{A_0}Z_0A_0 + 2L_{Z_0}A_1,$$ 

$$L_{A_0}\tilde{A}_2 = 0,$$ 

where the vector fields $Z_0$ and $\tilde{A}_1$ are given from the previous step. If after $(k - 1)$ steps, we have the vector fields $Z_0, \ldots, Z_{k-2}$ and $\tilde{A}_1, \ldots, \tilde{A}_{k-1}$, then on the $k$-th step
we have to find the solutions \( Z = Z_{k-1} \) and \( \bar{W} = \tilde{A}_k \) of equations (1.2.36),(1.2.37), where

\[
W = A_k + R_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\}.
\]

In the context of averaging method, equation (1.2.36) is called a homological equation, [5]. The solvability conditions of equations (1.2.36),(1.2.37) clearly depends on the properties of the unperturbed vector field \( A_0 \).

By Proposition 1.2.7 and Proposition 1.1.2, we have the following facts.

**Corollary 1.2.8** Suppose that homological equations (1.2.31),(1.2.32) are solvable on an open domain \( N_0 \subseteq M \) for \( k = 1, ..., N \) and let \( Z_0, ..., Z_{N-1} \) and \( \tilde{A}_1, ..., \tilde{A}_N \) be solutions. Then, for every open domain \( N \subseteq N_0 \) with compact closure, formula (1.2.35) defines a near identity transformation which is well defined on \( N \) and takes the \( \varepsilon \)-dependent vector field \( A_\varepsilon \) into normal form of order \( O(\varepsilon^N) \) on \( N \).

**Corollary 1.2.9 (Normalization of first order)** Let \( A_\varepsilon(x) \) be an \( \varepsilon \)-dependent vector field with Taylor expansion at \( \varepsilon = 0 \) is \( A_\varepsilon = A_0 + \varepsilon A_1 + O(\varepsilon^2) \). If there exists a vector field \( Z_0 \) such that

\[
(a) \text{ the flow } Fl_{Z_0} \
(b) \text{ satisfies on } N \text{ the equation } \mathcal{L}_{A_0} (\mathcal{L}_{A_0} Z_0 - A_1) = 0. \tag{1.2.39}
\]

Then, the near identity transformation \( \Phi_\varepsilon = Fl_{Z_0} \) sends the vector field \( A_\varepsilon \) to normal form of first order on \( N \) relative to \( A_0 \), that is, \( \Phi_\varepsilon^* A_\varepsilon = A_0 + \tilde{A}_1 + O(\varepsilon^2) \), where \( \tilde{A}_1 = A_1 - \mathcal{L}_{A_0} Z_0 \).

1.2.5 The Hamiltonian case

Here, we shall express the homological equation and recursive formulas (1.2.15) in terms of the Poisson bracket.

Recall that a Poisson bracket on a smooth manifold \( M \) is a \( \mathbb{R} \)-bilinear antisymmetric operation \( \{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \) compatible with the pointwise product of smooth functions by the Leibnitz rule and satisfying the Jacobi identity,

\[
\{F, GH\} = \{F, G\} H + \{F, H\} G, \tag{1.2.40}
\]

\[
\mathfrak{S}_{(F,G,H)} \{F, \{G, H\}\} = 0,
\]

where \( \mathfrak{S} \) denotes the cyclic sum.

The pair \((M,\{,\})\) is called a Poisson manifold and \((C^\infty(M),\{,\})\) is a Lie algebra. For every \( H \in C^\infty(M) \), we define the adjoint operator \( \text{ad}_H : C^\infty(M) \to C^\infty(M) \) given by \( \text{ad}_H (\cdot) = \{H, \cdot\} \). A smooth vector field \( X \) on a Poisson manifold \((M,\{,\})\) is said to be Hamiltonian relative to the Poisson bracket \( \{,\} \) if there exists a function \( H \in C^\infty(M) \) such that the Lie derivative along \( X \) coincides with the adjoint operator of \( H \),

\[
\mathcal{L}_X = \text{ad}_H. \tag{1.2.41}
\]
By the Leibnitz identity (1.2.40), every function \( H \in C^\infty(M) \) admits a unique Hamiltonian vector field in (1.2.41) which is denoted by \( X = X_H \). In local coordinates, Hamiltonian dynamical system generated by \( X_H \) is written in the bracket form by
\[
\dot{x}^i = \{H, x^i\}, \quad i = 1, 2, \ldots, 2n.
\]
The set \( \text{Ham}(M) \) of all Hamiltonian vector fields is a Lie subalgebra in \( \mathfrak{X}(M) \) and the correspondence \( H \mapsto X_H \) is a Lie algebra homomorphism,
\[
[X_{H_1}, X_{H_2}] = X_{\{H_1, H_2\}}.
\]
whose kernel is just \( \text{Casim}(M) \). A vector field \( P \) on the Poisson manifold \( M \) is said to be an infinitesimal Poisson automorphism (or, a Poisson vector field) if its Lie derivative is a derivation of the Poisson algebra \( (C^\infty(M), \{,\}) \),
\[
\mathcal{L}_P \{F_1, F_2\} = \{\mathcal{L}_P F_1, F_2\} + \{F_1, \mathcal{L}_P F_2\},
\]
for any \( F_1, F_2 \in C^\infty(M) \). It is clear that every Hamiltonian vector field is Poisson. The space of all Poisson vector fields form a Lie algebra, denoted by \( \text{Poiss}(M) \). It follows from (1.2.42) that
\[
[P, X_H] = X_{\mathcal{L}_P H}
\]
for any \( P \in \text{Poiss} (M) \) and \( H \in C^\infty(M) \). This property says that \( \text{Ham}(M) \) is an ideal of \( \text{Poiss}(M) \).

The Poisson bracket is called nondegenerate if every Casimir function \( K \in \text{Casim}(M) \) is a constant function. In this case there exists a unique nondegenerate closed 2-form \( \sigma \) on \( M \), which is compatible with Poisson bracket by the condition
\[
\sigma(X_{F_1}, X_{F_2}) = \{F_1, F_2\}.
\]
The pair \( (M, \sigma) \), where \( \sigma \) is a nondegenerate closed 2-form, is called a symplectic manifold. In terms of the symplectic structure \( \sigma \), condition (1.2.41) tells that a vector field \( X \) is Hamiltonian if there exists \( H \in C^\infty(M) \) such that
\[
\mathbf{i}_X \sigma = -dH.
\]
Suppose that the \( \epsilon \)-dependent vector field \( A_\epsilon \) is Hamiltonian relative to \( H_\epsilon = H_0 + \epsilon H_1 + \ldots \),
\[
A_\epsilon = X_{H_\epsilon} = X_{H_0} + \epsilon X_{H_1} + \ldots
\]
If \( Z_0 = X_{G_0}, \ldots, Z_{k-2} = X_{G_{k-2}} \) are Hamiltonian vector fields of functions \( G_0, \ldots, G_{k-1} \in C^\infty(M) \), Proposition 1.2.6 implies that the vector field
\[
R_{k-1}^D\{X_{G_0}, \ldots, X_{G_{k-2}}; X_{H_0}, \ldots, X_{H_{k-1}}\} = X_{R_{k-1}}
\]
where the vector fields \( R_{k-1}^D \), described in Proposition 1.2.3, are also Hamiltonian relative to the functions \( R_{k-1} = R_{k-1}^D\{G_0, \ldots, G_{k-2}; H_0, \ldots, H_{k-1}\} \). In particular,
\[
R_0 = 0, \quad R_1 = 2\{G_0, H_1\{G_0, H_0\}\},
\]
and

\[ \mathcal{R}_2 = \{G_0, 3H_2 + \{G_0, 3H_1 + \{G_0, H_0}\}\} + \{G_1, 2H_0\} + \{G_1, 3H_1 + \{G_0, H_0\}\}. \]

An advantage of the Hamiltonian case is that the problem (1.2.36),(1.2.37) can be reduced to the study of homological equations for functions. Assume that vector fields \( A_0 = X_{H_0}, W = X_{F} \) are Hamiltonian on a Poisson manifold \((M, \{\})\). If there exist smooth functions \( G \) and \( \bar{F} \) satisfying the equations

\[
\{H_0, G\} = F - \bar{F}, \tag{1.2.44}
\]
\[
\{H_0, \bar{F}\} = 0, \tag{1.2.45}
\]

then the Hamiltonian vector fields \( Z = X_{G} \) and \( \bar{W} = X_{\bar{F}} \) are solution to the problem (1.2.36),(1.2.37).

Consider the following generalization of the Hamiltonian case. Suppose we have a perturbed vector field of the form

\[
A_\varepsilon = P + \varepsilon X_{H_1} + \frac{\varepsilon^2}{2} X_{H_2} + \ldots,
\]

where \( P \) is a Poisson vector field on \( M \), which plays the role of the unperturbed vector field. But the perturbation vector field remains Hamiltonian corresponding to an \( \varepsilon \)-dependent function \( \varepsilon H_1 + \frac{\varepsilon^2}{2} H_2 + \ldots \). Then, \( A_0 = P, W = X_{H_1} \). Putting again \( Z = X_{G} \) and \( \bar{W} = X_{\bar{F}} \) into (1.2.36),(1.2.37) and using (1.2.43), we get the following equations for functions \( G, \bar{F} \):

\[
\mathcal{L}_P G = H_1 - \bar{F},
\]
\[
\mathcal{L}_P \bar{F} = 0.
\]

### 1.3 Normalization Transformations Around Invariant Submanifolds

According to Proposition 1.2.7, the normalization of an \( \varepsilon \)-dependent vector field can be proceeded in two steps: (a) solving homological type problems (1.2.36),(1.2.37) and then (b) studying the domain of definition of the flow of time-dependent vector field (1.2.34).

Here, we consider a class of perturbed systems on a manifold \( M \) (not necessarily compact) for which condition (b) of Corollary 1.2.7 holds.

Suppose we are given an \( \varepsilon \)-dependent vector field \( A_\varepsilon \) on \( M \) which has an invariant submanifold \( S \subset M \) (\( \dim S < \dim M \)) and the restriction of \( A_\varepsilon \) to \( S \) does not depend on \( \varepsilon \),

\[
A_\varepsilon(x) \in T_x S \quad \forall x \in S, \varepsilon \in \mathbb{R}, \tag{1.3.1}
\]
\[
v \overset{\text{def}}{=} A_\varepsilon|_S \quad \text{is independent of } \varepsilon. \tag{1.3.2}
\]

In terms of the coefficients \( A_k \) of Taylor expansion (1.2.4) these conditions can be reformulated as follows: the submanifold \( S \) is invariant with respect to the flow of
the unperturbed vector field $A_0$ and the perturbation vector fields $A_1, A_2, \ldots$ vanish at $S$, that is,

\begin{align}
A_0(x) &\in T_xS \quad \forall x \in S, \quad (1.3.3) \\
A_k|_S &= 0 \quad (k = 1, 2, \ldots) \quad (1.3.4)
\end{align}

**Definition 1.3.1** We say that a near identity transformation $\Phi_\varepsilon : N \rightarrow M \ (\varepsilon \in (-\delta, \delta))$ is compatible with submanifold $S \subset M$ (or, shortly $S$-compatible) if $\Phi_\varepsilon$ is a diffeomorphism from $N$ onto another open neighborhood of $S$ in $M$

\[ S \subset \Phi_\varepsilon(N) \quad (1.3.5) \]

and

\[ \Phi_\varepsilon|_S = \text{id} \quad (1.3.6) \]

for all $\varepsilon \in (-\delta, \delta)$.

An important class of $S$-compatible near identity transformations can be obtained in the following way.

**Lemma 1.3.1** Let $Z_\varepsilon$ be an $\varepsilon$-dependent vector field on $M$ vanishing at the submanifold $S \subset M$, $Z_\varepsilon|_S = 0$, $\forall \varepsilon \in \mathbb{R}$. Then, for every $\delta > 0$ there exists an open neighborhood $N = N_\delta$ of $S$ in $M$ such that the flow $\text{Fl}^\varepsilon_{Z_\varepsilon}$ of $Z_\varepsilon$ is well defined on $N$ for all $\varepsilon \in (-\delta, \delta)$. Moreover, $\text{Fl}^\varepsilon_{Z_\varepsilon} : N \rightarrow M$ is a near identity transformation compatible with $S$.

**Proof.** We fix $\delta > 0$. By the flow box theorem, for every $\xi \in S$ there exists an open neighborhood $U_\xi$ of $\xi$ on $M$ such that the flow $\text{Fl}^\varepsilon_{Z_\varepsilon}$ is well defined on $U_\xi$ for all $\varepsilon \in (-\delta, \delta)$. Let $N_\delta \overset{\text{def}}{=} \bigcup_{\xi \in S} U_\xi$ be an open neighborhood of $S$. Thus, $\text{Fl}^\varepsilon_{Z_\varepsilon} : N_\varepsilon \rightarrow M$ is well defined for all $\varepsilon \in (-\delta, \delta)$ and a diffeomorphism onto its image. Since $\text{Fl}^\varepsilon_{Z_\varepsilon}$ vanishes at $S$, we have that $\text{Fl}^\varepsilon_{Z_\varepsilon}(\xi) = \xi$ for all $\varepsilon$. It follows that $\text{Fl}^\varepsilon_{Z_\varepsilon}|_S = \text{id}$ and $S \subset \text{Fl}^\varepsilon_{Z_\varepsilon}(N)$, for $\varepsilon \in (-\delta, \delta)$. Therefore, $\text{Fl}^\varepsilon_{Z_\varepsilon}$ is a $S$-compatible near identity transformation.

**Definition 1.3.2** We say that the perturbed vector field $A_\varepsilon$ satisfying (1.3.1),(1.3.2) admits a normalization of order $K$ around the invariant submanifold $S$ if there exists a $S$-compatible near identity transformation $\Phi_\varepsilon$ such that the pull-back $(\Phi_\varepsilon)^* A_\varepsilon$ is in normal form (1.1.11), (1.1.12).

We denote by $C^\infty_S(M)$ and $\chi_S(M)$ the Lie subalgebra of smooth functions vector fields on $M$ vanishing at $S$. It is clear that the Lie derivative $\mathcal{L}_{A_0}$ along $A_0$ leaves invariant these subalgebras. Moreover, $\chi_S(M)$ is a $C^\infty_S(M)$-module.

If $Z_0, \ldots, Z_{k-2}$ are vector fields vanishing at $S$. Then, the vector field

\[ R^D_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\}, \]

where the vector fields $R^D_{k-1}$ are described in Proposition 1.2.3, also vanishing at $S$.

Therefore, if there exists $S$-compatible normalization then the vector field $(\Phi_\varepsilon)^* A_\varepsilon$ automatically vanish at $S$, that is,

\[ \tilde{A}_1|_S = \ldots = \tilde{A}_K|_S = 0. \]
Proposition 1.3.2 Let $A_\varepsilon = A_0 + \ldots + \varepsilon^K \frac{K!}{K!}$ be an $\varepsilon$-dependent vector field vanishing at $S$. Assume that there exist vector fields $Z_0, \ldots, Z_{N-1} \in \mathfrak{X}_S(M)$ and $\tilde{A}_1, \ldots, \tilde{A}_K \in \mathfrak{X}_S(M)$ satisfying the recurrent equations

$$\mathcal{L}_{A_0} Z_{k-1} = A_k - \tilde{A}_k + R_{k-1}\{Z_0, \ldots, Z_{k-2}; A_0, \ldots, A_{k-1}\} \quad (1.3.7)$$

$$\mathcal{L}_{A_0} \tilde{A}_k = 0 \quad (1.3.8)$$

for $k = 1, \ldots, N$. Then, $A_\varepsilon$ admits normalization of order $K$, where the $S$-compatible near identity transformation $\Phi_\varepsilon$ is defined as the flow of the time-dependent vector field $Z_\varepsilon = Z_0 + \varepsilon Z_1 + \varepsilon^2 \frac{K!}{K!} Z_{K-1} \in \mathfrak{X}_S(M)$,

$$\Phi_\varepsilon = F^{(\varepsilon)}_{Z^{(K)}}. \quad (1.3.9)$$